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Adding flavor to Dijkgraaf-Vafa

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Abstract: We study matrix models related via the correspondence of Dijkgraaf and Vafa to supersymmetric gauge theories with matter in the fundamental. As in flavorless examples, measure factors of the matrix integral reproduce information about R-symmetry violation in the field theory. The models, studied previously as models of open strings, exhibit a large-M phase transition as the number of flavors is varied. This is the matrix model's manifestation of the end of asymptotic freedom. Using the relation to a quiver gauge theory, we extract the effective glueball superpotential and Seiberg-Witten curve from the matrix model.

Keywords: 1/N Expansion, Matrix Models, D-branes, String Duality.

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Contents

1. Introduction and summary

Dijkgraaf and Vafa have found that matrix integrals compute all of the holomorphic information in $\mathcal{N}=1$ gauge theories with classical gauge groups [1, 2, 3]. In the interest of understanding their proposal better, we will look at the matrix model that they prescribe for $\mathcal{N}=2$ gauge theory with N_f fundamental hypers mass-deformed to $\mathcal{N}=1$ by a tree level superpotential for the adjoint chiral field Φ, and masses for the squarks. These field theories, studied in [4], exhibit fascinating behavior as a function of the number of flavors.

Such models arise in string theory in a number of ways. The $\mathrm{U}(N)$ theories arise by probing the "canonical example" of Dijkgraaf-Vafa with D5-branes on a noncompact curve of the Calabi-Yau (CY) [5]. Flavorful theories with real gauge groups arise on N D3-brane probes of D7-brane configurations [6, 7, 8, 9]; this fact was recently employed to add holes to the BMN worldsheet [10]. We will instead use it to poke holes in the random surfaces described by the Dijkgraaf-Vafa matrix integrals.

In particular, for $N=1$, the theories with $N_f \leq 4$ (with $\mathrm{Sp}(2N)=\mathrm{SU}(2)$ gauge group) arise on a D3-brane probe of a resolved D_4 singularity of F-theory. The theory with $N_f=4$ massless flavors is obtained by creating the D_4 singularity; in this case the dilaton is constant and the gauge theory is scale-invariant. This brings us to the question we would like to answer: How is this dependence on N_f manifested in the Dijkgraaf-Vafa matrix model?

We are therefore led to consider a matrix integral of the form

$$Z(g_k, m_\alpha, M) = \int d\Phi dQ d\tilde{Q} \exp\left(-W_0(\Phi) + \tilde{Q}_\alpha \Phi Q^\alpha - \sum_{\alpha=1}^{M_f} \tilde{Q}_\alpha Q^\alpha m_\alpha\right) \tag{1.1}$$

where

$$W_0(\Phi) = \sum_{k=1}^{n+1} g_k \mathrm{tr}\ \Phi^k\,,$$

Φ is a complex $M \times M$ matrix, Q is a complex $M \times M_f$ rectangle, and $\tilde{Q}$ is a complex $M_f \times M$ rectangle. An arbitrary meson source $\tilde{Q}_\alpha m^\alpha_\beta Q^\beta$ can be brought into this diagonal form by rotating the Qs. As in the work of Dijkgraaf and Vafa, these are to be thought of as line integrals over matrices of complex numbers.

Without using very much technology, we can study in detail the model which arises upon integrating out the fundamentals. This generates a $\ln(m - \Phi)$ potential for Φ. The model with this addition to the potential can still be solved by the method of BIPZ [11], and actually was studied in this way as a model for open strings by Kazakov [12].[1] The coefficient of the log is $\gamma \equiv N_f/N$, and we will show that the cuts close up when γ is chosen so that the gauge theory can be conformal. Related by supersymmetry to this statement is the fact that the R-symmetry becomes non-anomalous for this choice of N_f/N. This fact can be detected in the matrix model through the dependence of the measure on γ. Finally, we extract the glueball superpotential and Seiberg-Witten curve from the large-M solution to the matrix model.

1.1 M's and N's

One of the more mysterious aspects of the Dijkgraaf-Vafa prescription, at least to the author, is the disjunction between the number of colors N of the gauge theory, and the number of colors M of the matrix model (which plays the role of the glueball superfield). The addition of N_f flavors to the gauge theory only complicates this issue. We will introduce a number of flavors M_f in the matrix model which is again not the same as the corresponding number in the gauge theory. We will, however, identify the ratio

$$\gamma \equiv \frac{N_f}{N} = \frac{M_f}{M}\,;$$

This will be the parameter of interest in our study of the matrix model. We take this as part of the prescription, but (thinking of γ as the weight with which holes in the random surface contribute) one which is again motivated by topological string duality [16]. Specifically, this is the usual Dijkgraaf-Vafa limit for a two-node quiver gauge theory [2] which reduces to the flavorful theory when the dynamics of the gauged flavor group are frozen out. It is from this perspective that we will be able to extract the superpotential.

[1] Related models were also studied in [13, 14], and in particular in [15] where the critical behavior at $\gamma = 2$ was explored.

1.2 Related work

Matrix models with fundamentals in this context are mentioned in a footnote in [1]. They also make an appearance in the very recent [17, 18]. Other work on understanding and extending the Dijkgraaf-Vafa proposal includes [19]–[27].

2. Flavorful matrix models

To begin, we write down the matrix integral with naive couplings and fields marked with hats:

$$Z = \int d\hat{\Phi} d\hat{Q} d\hat{\tilde{Q}} \exp\left(-\hat{W}_0(\hat{\Phi}) + \hat{\tilde{Q}}_\alpha \hat{\Phi} \hat{Q}^\alpha - \sum_{\alpha=1}^{M_f} \hat{\tilde{Q}}_\alpha \hat{Q}^\alpha \hat{m}_\alpha \right) \tag{2.1}$$

with

$$\hat{W}_0(\hat{\Phi}) \equiv \hat{g}_1 \mathrm{tr}\ \hat{\Phi} + \hat{g}_2 \mathrm{tr}\ \hat{\Phi}^2 + \cdots .$$

α is a flavor index; color indices will be denoted $a, b, \ldots$ These hatted fields and couplings will be related to those which should have finite large-M limits (which lack hats) by an M-dependent rescaling. These hatless variables are chosen below so that there is a well-peaked saddle point of the Φ integral, the location of which is M-independent. We will observe that the precious

$$\frac{1}{2} M^2 \ln M$$

term in the matrix model free energy, which is derived from the inverse volume of the matrix model gauge group, can also be detected by such a propitious field rescaling.

2.1 Integrating out the flavor

For the moment, we are interested in the regime of couplings where the quark masses, m_α, are much bigger than the bare mass of the adjoint in $W_0(\Phi)$. In this regime, we integrate out the fast Q modes at fixed Φ to get an effective potential for Φ. The integrals over Q_α are M_f independent gaussian integrals. This gives

$$\begin{aligned} Z &= \int d\hat{\Phi} e^{-\hat{W}_0(\hat{\Phi})} \prod_{\alpha=1}^{M_f} \det_{ab}{}^{-1} \left(\hat{\Phi}^a_b - \hat{m}_\alpha \delta^a_b \right) \\ &= \int d\hat{\Phi} \exp\left(-\hat{W}_0(\hat{\Phi}) - \sum_{\alpha=1}^{M_f} \mathrm{tr}\ \ln(\hat{\Phi} - \hat{m}_\alpha 1) \right) . \end{aligned} \tag{2.2}$$

Setting all of the masses equal to m for simplicity, this is

$$Z = \int d\hat{\Phi} \exp\left(-\hat{W}_0(\hat{\Phi}) - M_f \mathrm{tr}\ \ln(\hat{\Phi} - \hat{m}) \right) . \tag{2.3}$$

A matrix integral very similar to (2.3) was studied by [12, 13, 15, 14] as a discretization of an open string worldsheet. In this model, the counterpart of M_f is the weight accompanying a hole insertion. The logarithmic potential was chosen to reproduce a sum over discretizations of the worldsheet boundaries, with equal weight for arbitrary numbers of segments of the boundary.

Other than a relabeling of couplings, the difference between our model and that of Kazakov is that the logarithmic potential term of [12] is

$$\mathrm{tr}\ \ln(m-\varphi^2)\,.$$

This is the potential that would arise if the $\mathcal{N}=2$ superpotential were $\tilde{Q}\Phi^2 Q$ instead of $\tilde{Q}\Phi Q$. The field redefinition $\Phi=\varphi^2$ required to relate the two integrals directly introduces a jacobian factor which adds a term

$$\frac{1}{2}\mathrm{tr}\ \ln\Phi$$

to the potential. From the calculation (2.2) above we see that this is the same as the effect of adding $M/2$ *massless* hypers. We will find it convenient to solve the integral (2.3) directly. The qualitative behavior we find is the same as that found in [12].

2.2 The continuum

In order to proceed, diagonalize the matrix Φ as $\Phi=\mathcal{U}\mathcal{D}\mathcal{U}^\dagger$ with

$$\mathcal{D}\equiv\mathrm{diag}(\lambda_1,\ldots,\lambda_M)\,,$$

and $\mathcal{U}$ unitary. These eigenvalues are normalized as in [11]. By the magic of logarithms, the integrand of (2.3) does not depend on the angular $\mathcal{U}$ variables. Their integration produces the Vandermonde determinant

$$\Delta(\lambda)=\prod_{a<b}(\lambda_a-\lambda_b)^2\,.$$

The integral becomes

$$Z=\int\prod_a d\lambda_a\Delta(\lambda)\exp\sum_{a=1}^{M}\left(-\hat{W}_0(\lambda_a)+\sum_{\alpha=1}^{M_f}\ln(\lambda_a-\hat{m}_\alpha)\right)\,. \qquad (2.4)$$

At this point, it is convenient to introduce a continuum in the space of colors. Let

$$\lambda_a=\sqrt{M}\lambda\left(\tilde{a}=\frac{a}{M}\right),\qquad 1=\int_0^1 d\tilde{a}=\frac{1}{M}\sum_{a=1}^{M}\,.$$

The eigenvalue density

$$\rho(\mu)=\frac{d\tilde{a}}{d\lambda}$$

is normalized to

$$\int d\mu\rho(\mu)=1\,. \qquad (2.5)$$

2.2.1 Large M scaling

We now introduce the promised variables without hats:

$$\begin{aligned}
&\hat{g}_2 = \frac{g_2}{S}\,, \qquad \hat{g}_3 = \frac{g_3}{S\sqrt{M}}\,, \qquad \ldots, \qquad \hat{g}_k = \frac{g_k}{g_s M^{k/2}} = \frac{g_k}{S M^{k/2-1}} \\
&\hat{m}_i = \sqrt{M} m_i \\
&\hat{\Phi} = \sqrt{M}\Phi\,, \qquad \hat{Q} = \frac{1}{M^{1/4} g_s^{1/2}} Q\,, \qquad \hat{\tilde{Q}} = \frac{1}{M^{1/4} g_s^{1/2}} \tilde{Q}\,.
\end{aligned} \tag{2.6}$$

Here we have finally introduced the quantity $S \equiv g_s M$, which is fixed in the large-M limit. Note that these rescalings are closely related to those made on dimensional grounds in the stringy realization of the gauge theory [5]. Plugging these into (2.1), we find that the resulting λ integral is of the form

$$Z = \int D\lambda(\tilde{a}) \exp\left(\frac{1}{g_s^2}\mathcal{F}_0[\lambda] + \mathcal{C}(M)\right) \tag{2.7}$$

with $\mathcal{C}(M)$ independent of λ and

$$\begin{aligned}
\mathcal{F}_0[\lambda] &= S^2 \int_0^1 d\tilde{a} \int_0^1 d\tilde{b} \ln(\lambda(\tilde{a}) - \lambda(\tilde{b})) - S \int_0^1 d\tilde{a}\ W_0(\lambda(\tilde{a})) - S^2\gamma \int_0^1 d\tilde{a} \ln(\lambda(\tilde{a}) - m) \\
&= S^2 \int d\lambda \int dz \rho(\lambda)\rho(z) \ln(\lambda - z) - S \int d\lambda \rho(\lambda) W_0(\lambda) - S^2\gamma \int d\lambda \rho(\lambda) \ln(\lambda - m).
\end{aligned} \tag{2.8}$$

Here $W_0(\lambda) = g_1\lambda + g_2\lambda^2 + \cdots$. The crucial feature of (2.7) is that $\mathcal{F}_0[\lambda]$ is independent of M. This is the normalization used by Dijkgraaf and Vafa; the matrix action is

$$-\frac{1}{g_s}\left(W_0(\Phi) + \tilde{Q}\Phi Q - m\tilde{Q}Q\right) \tag{2.9}$$

Now we return to the "constant g-independent term" [11] $\mathcal{C}(M)$. This field- and coupling-independent term was not relevant for previous applications of matrix integrals. It is

$$\begin{aligned}
\mathcal{C}(M) &= \frac{1}{2} M^2 \ln M - \frac{1}{4} M_f M \ln M \\
&= (2-\gamma)\frac{1}{4} M^2 \ln M\,.
\end{aligned} \tag{2.10}$$

This reproduces the leading M-dependence of the log of the inverse volume of $\mathrm{U}(M)$ [28] in the field normalization we are using. Further, it provides the "entropy factor" arising from the flavor integrals[2] The Dijkgraaf-Vafa prescription relates the matrix model free energy to the prepotential of the gauge theory. In parallel with the discussion of [1] this term leads to the following contribution to the effective superpotential of the gauge theory:

$$W_{\text{eff}}(S) = (2-\gamma) N S \ln \frac{S}{\Lambda^3}\,, \tag{2.11}$$

up to linear terms in S which are independent of the cutoff scale Λ_0. The prefactor of this Veneziano-Yankielowicz superpotential is proportional to the anomaly in the $\mathrm{U}(1)_R$ current of the field theory. The introduction of N_f flavors in the fundamental modifies this from $2N$ to $(2-\gamma)N$. It is gratifying that this is reproduced by the simple matrix integral.

[2] I am grateful to Nissan Itzhaki for comments on this point.

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2.3 Solution at large M

In terms of the variables normalized to have a finite large-M limit, the saddle point equation is

$$\frac{1}{S}W_0'(\lambda) + \frac{\gamma}{\lambda - m} = 2 \fint d\mu \frac{\rho(\mu)}{\lambda - \mu} . \tag{2.12}$$

Rewrite this equation as

$$\frac{1}{S}W_0'(\lambda) = 2 \fint d\mu \frac{\rho_0(\mu)}{\lambda - \mu} \tag{2.13}$$

where

$$\rho_0(\mu) \equiv \rho(\mu) - \frac{\gamma}{2}\delta(\mu - m) , \tag{2.14}$$

or more generally in the case of arbitrary masses

$$\rho_0(\mu) = \rho(\mu) - \gamma \frac{1}{2M_f} \sum_{\alpha=1}^{M_f} \delta(\mu - m_\alpha) .$$

Therefore the eigenvalue density ρ_0 satisfies the same integral equation as that of the theory *without* flavors, with the modified boundary condition

$$\int d\mu \rho_0(\mu) = \int d\mu \rho(\mu) - \int d\mu \sum_{\alpha=1}^{M_f} \frac{\gamma}{2M_f} \delta(\mu - m_\alpha) = \frac{1}{2}(2 - \gamma) .$$

This integral is to be performed over the real μ line.

For simplicity, let us consider the case of a cubic superpotential, in the case of a single cut, i.e. choose the potential W_0 to have a single critical point. Placing the cut at $[2a, 2b]$, the BIPZ solution for the resolvent

$$\omega_0(\lambda) = \int_{2a}^{2b} d\mu \frac{\rho_0(\mu)}{\mu - \lambda} \tag{2.15}$$

of the theory without flavors determines the solution of (2.12). This is [11]

$$S\omega_0(z) = 2g_2 z + 3g_3 z^2 - (2g_2 + 3g_3(a+b) + 3g_3 z)\sqrt{(z-2a)(z-2b)} . \tag{2.16}$$

The conditions determining the positions a, b of the ends of the cut are determined by the behavior of (2.16) at $z \to \infty$. For the model with flavor, these are

$$3g_3(b-a)^2 + 2(a+b)(2g_2 + 3g_3(a+b)) = 0 \tag{2.17}$$

$$\frac{1}{S}(b-a)^2(2g_2 + 6g_3(a+b)) = 2 - \gamma . \tag{2.18}$$

Up to coupling redefinitions, these differ from eqs. (46) of [11] only by the replacement $2 \mapsto 2 - \gamma$ in the second condition. Note that they do not depend on m.

2.3.1 Healing of cuts

Assume there is a stable vacuum at the origin, and place all of the eigenvalues there, e.g. consider $g_k = 0$, $k \geq 3$, $g_2 > 0$. From (2.17) and (2.18) we immediately see that when $\gamma \to 2$, $b - a \to 0$. That is, the cut closes up. Beyond $\gamma = 2$, the cut at the origin moves into the complex plane. In the context of a hermitean matrix integral [12] this was interpreted as a large-M phase transition beyond which the theory lacked a stable solution. However, when the saddle point we are studying is that of a holomorphic line integral, this is not so catastrophic. In fact, the cut surrounding the unstable extrema of W always extend into the imaginary direction of λ.

In the matrix models for confining gauge theories, the size of the cuts which hold the eigenvalues goes like the IR scale Λ of the field theory. The closing of the cuts is a signal that the theories become scale invariant, and then no longer asymptotically free as N_f passes through $2N$.

We can see the corresponding effect for more general tree-level superpotentials as follows. As a consequence of the saddle equation (2.13), the resolvent

$$\omega_0(x) = \int d\lambda \frac{\rho_0(\lambda)}{x - \lambda}$$

of the $\gamma = 0$ theory satisfies at large M an algebraic equation of the form e.g. [29]

$$\omega_0(x)^2 + \frac{1}{S}\omega_0(x) W_0'(x) + \frac{1}{4S^2} f_0(x) = 0\,. \tag{2.19}$$

Here $f_0(x)$ is a degree $n-1$ polynomial in x, which should be thought of as a function of $S_i = g_s M_i$, with M_i the number of eigenvalues in the ith cut. This polynomial f_0 differs from the one in the $\gamma = 0$ solution through its dependence on the locations of the cuts. Note that the resolvent of the theory with flavor is related to ω_0 by the addition of a pole term

$$\omega(x) = \omega_0(x) + \frac{\gamma}{x - m}\,. \tag{2.20}$$

The remainder term, f_0, can be written [29]

$$f_0(z) = 4S \int dw\ \rho_0(w) \frac{W_0'(w) - W_0'(z)}{w - z}\,;$$

this is a polynomial of degree $n-1$ in z, $f_0(z) = \sum_{k=1}^{n-1} b_k z^k$. The fact that $\int dw\, \rho_0(w) = 2 - \gamma$ then implies that as $\gamma \to 2$, the leading term, b_{n-1}, vanishes. In more detail, (2.19) is a quadratic equation for ω_0, which therefore has the solution

$$2S\omega_0(z) = W_0'(z) \pm \sqrt{(W_0'(z))^2 - f_0(z)}\,.$$

From this expression we learn that

$$b_{n-1} = -2(2-\gamma)(n+1) g_{n+1} S\,.$$

But, we also know from [5] and from [30] that

$$b_{n-1} = -4(n+1) g_{n+1} \frac{\partial W_{\rm eff}}{\partial \ln \Lambda^{2N}}\,.$$

Therefore, we see from the matrix model that the superpotential is independent of the IR scale of the gauge theory when $\gamma \to 2$. If $m = 0$, all the cuts are healed, in accord with the fact that in that case scale invariance is never broken.

2.4 The Seiberg-Witten curve and glueball superpotential

In this subsection we will extract the glueball superpotential from the free energy of the matrix model, and the Seiberg-Witten (SW) curve of the gauge theory from the loop equation.[3]

In the canonical example of Dijkgraaf-Vafa, the free energy of the matrix model determines the effective glueball superpotential of the field theory via

$$W_{\rm eff}(S_i) = \sum_i N_i \frac{\partial \mathcal{F}_0}{\partial S_i} - 2\pi i \tau S \tag{2.21}$$

with $\mathcal{F}_0(S_i) = g_s^2 \ln Z$. Thinking of the flavors as arising from the $\mathcal{N} = 2$ quiver with two nodes of respective ranks N and N_f in the limit that the coupling of the flavor group vanishes, this formula is modified in our case to[4]

$$W_{\rm eff} = -2\pi i\tau S + \sum_i N_i \frac{\partial \mathcal{F}_0}{\partial S_i} + N_f \frac{\partial \mathcal{F}_0}{\partial S_f} \tag{2.22}$$

where $S_f = g_s M_f$. Because the flavor gauge coupling is zero, we do not include the dynamics of the corresponding matrix variables, and treat them as a background. (2.22) has been written for the case where all of the flavor branes are coincident (i.e. the masses for the flavors are identical), but the generalization is clear.

A convenient way [37] to compute $\partial\mathcal{F}_0/\partial S_i$ is by varying the eigenvalue density according to

$$\rho(z) \mapsto \rho(z) + \delta S_i \frac{1}{S}\delta(z - e_i)$$

with e_i some fixed point inside the ith cut. From (2.8) evaluated in the saddle, we find

$$\frac{\partial \mathcal{F}_0}{\partial S_i} = \int_{e_i}^{\Lambda_0} dx \left[W_0'(x) - S\left(2\int d\lambda \frac{\rho(\lambda)}{x-\lambda} - \frac{\gamma}{x-\lambda}\right)\right]_{\rho(\lambda)=\rho_0(\lambda)+\frac{\gamma}{2}\delta(\lambda-m)} .$$

We have ignored the additive constant, and Λ_0-independent terms linear in S, which can be absorbed in the bare coupling, τ. This expression is identical to that for the case without flavor; the contributions proportional to γ cancel each other. So we have

$$\frac{\partial \mathcal{F}_0}{\partial S_i} = \int_{B_i} dx W_0'(x) - 2S\int dw \rho_0(w)\int_{B_i} \frac{dx}{x-w} = \int_{B_i} dx \left(W_0'(x) + 2S\omega_0(x)\right) \tag{2.23}$$

where B_i is a contour running from the reference point e_i to the cutoff Λ_0. As in the case without flavor, we introduce y, the singular piece of the ($\gamma = 0$) resolvent, by completing the square in (2.19) [1]

$$y(x) = 2S\omega_0(x) + W_0'(x) .$$

[3] Please note that this subsection was rewritten for version three, after the appearance of [31]–[38] in which superpotentials were found using matrix models with a finite number of flavors.

[4] While this paper was being revised, a closely related observation was made independently in [39].

Then (2.23) becomes

$$\frac{\partial \mathcal{F}_0}{\partial S_i} = \int_{B_i} y dx\,. \tag{2.24}$$

Similarly, we can determine the value of the new term in (2.22) by varying

$$\rho(z) \mapsto \rho(z) + \frac{\delta\gamma}{2}\,\delta(z - m_0)\,; \tag{2.25}$$

m_0 is the mass at which we add the new flavor. The factor of two in (2.25) appears because of the corresponding factor in (2.14). Under this change,

$$\delta_\gamma \mathcal{F}_0 = \frac{1}{2}\left[-SW_0(m_0) + 2S^2 \int d\lambda\ \rho(\lambda)\ln(\lambda - m_0) - S^2\gamma\ln(m_0 - m)\right]_{\rho(\lambda)=\rho_0(\lambda)+\frac{\gamma}{2}\delta(\lambda-m)}. \tag{2.26}$$

Again up to irrelevant terms, this can be rewritten to give

$$\begin{aligned}
\frac{\partial \mathcal{F}_0}{\partial S_f} &= \frac{1}{2}\left[-\int_{m_0}^{\Lambda_0} dx W_0'(x) + 2S\int_{m_0}^{\Lambda_0} dx \int \frac{d\lambda}{\lambda - x}\Big(\rho_0(\lambda) + \frac{\gamma}{2}\delta(\lambda - m)\Big) + \gamma S \int_{m_0}^{\Lambda_0} \frac{dx}{x - m}\right] \\
&= -\frac{1}{2}\int_{m_0}^{\Lambda_0} dx\,\big(2S\omega_0(x) + W_0'(x)\big) = -\frac{1}{2}\int_{m_0}^{\Lambda_0} y dx\,.
\end{aligned}$$

We therefore find

$$W_{\text{eff}}(S_i) = \sum_i N_i \int_{B_i} y dx - \frac{1}{2} N_f \int_m^{\Lambda_0} y dx \tag{2.27}$$

with y defined by

$$y^2 = (W_0')^2 - f_0\,.$$

Here, the coefficients of $f_0(x) = \sum_k b_k(S_i)x^k$ are determined by the dynamical glueball fields S_i according to

$$S_i = \int_{A_i} y dx$$

where A_i is a contour encircling the ith cut. This is the formula for the effective superpotential predicted [5] from the physics of D5-branes on the generalized conifold.

Taking $N_i = 1$, minimization of this superpotential with respect to variations of the polynomial f_0 has been shown [40, 37, 30] using the methods of [41] to result in the correct Seiberg-Witten curve

$$y^2 = \prod_{a=1}^{N} (x - \phi_a)^2 + \Lambda^{2N - N_f}(x - m)^{N_f}\,, \tag{2.28}$$

where ϕ_a are the critical points of W_0.

3. Discussion and prospects

In this paper, we have focused on the regime of couplings $m_\Phi \ll m_Q$ (though of course we can still perform the gaussian integral over Q in the other regime) where the interesting observables involve the adjoint field, as in the case without flavors. It will be interesting to try to compute other observables involving light quarks. In the SU(N) version of these theories, these include baryon operators

$$Z[M,B,\tilde{B}] = \int d\Phi dQ d\tilde{Q} \exp\left(W_0(\Phi) - \tilde{Q}_\alpha \Phi Q^\alpha + \sum_\alpha \tilde{Q}_\alpha Q^\alpha m_\alpha\right) \times$$
$$\times \exp\left(\tilde{Q}^a_\alpha M^\alpha_\beta Q^\beta_a + B^{\alpha_1\ldots\alpha_N}\epsilon_{a_1\ldots a_N} Q^{\alpha_1}_{a_1}\cdots Q^{a_N}_{\alpha_N} + \tilde{B}^{\alpha_1\ldots\alpha_N}\epsilon_{a_1\ldots a_N}\tilde{Q}^{a_1}_{\alpha_1}\cdots\tilde{Q}^{a_N}_{\alpha_N}\right).$$

These baryon sources exist for $N_f \geq N$, corresponding to a value of γ at which we have not yet detected any change in behavior of the matrix model.

3.1 Beyond the transition

Consider the F-theory realization of the related symplectic models mentioned in the introduction (Sp(N) with $N_f \leq 4$ and an antisymmetric tensor). The flavor symmetry (which is SO(8) in the critical case) is the gauge symmetry on the D7-branes. Increasing N_f/N beyond the critical value is achieved by adding more D7-branes to the D_4 singularity. This is possible without destroying the triviality of the canonical bundle, and one obtains in this way collections of D7-branes with the exceptional series of gauge groups of rank up to 8. The D3 probe theories are then field theories with exceptional flavor symmetry e.g. [42]–[46].

It has become clear that the complex x-plane of Φ eigenvalues can be identified with the image of a fibration of a noncompact CY geometry. A cut which holds the eigenvalues in the large M solution is identified with the image in the x-plane of a three-cycle in this geometry (after a geometric transition induced by the flux generating $W_0(\Phi)$). It is therefore tempting to speculate that tuning γ past the critical value is the matrix model version of performing an extremal transition in the CY geometry, during which the three-cycle shrinks and one finds an even-dimensional cycle which can be resolved. The fact that a shrinking del Pezzo four-cycle in a CY realizes a theory with exceptional flavor symmetry leads to a clear candidate for the nature of this new direction.

3.2 Some remaining issues

1. Our calculations should extend to the case of real gauge groups, and in particular to the theories with extra tensor representations, arising from D3-brane probes of F-theory.

2. "uv" completions of the Seiberg-Witten curve can be seen from the matrix model. As explained in [26], including more of the "fractional branes" of the CY singularity allows one to determine an embedding of the SW curve in a threefold of a form such as

$$uv = F(x,y)\,.$$

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This is important, for example, because it will allow one to identify the resolution involved in the extremal transition proposed above.

3. The field theories obtained on 3-brane probes of F-theory exhibit S-duality. In a beautiful series of papers [21, 22, 23, 27], the S-duality of the $\mathcal{N}=4$ theory and its $\mathcal{N}=1$ deformations has been found via the solution of the corresponding matrix integral in [47]. The symplectic matrix model with the corresponding matter content should also have modular behavior in τ; it shares the feature with the $\mathcal{N}=4$ theory that the path integral over the matter cancels the Vandermonde for the adjoint matrix.

4. Kazakov [12] computes "average numbers of holes" and "average lengths of holes" in the random surfaces, from the large M solution to the matrix integral. These observables exhibit more detailed critical behavior than we have discussed thus far as γ, m, g_k are varied. The transition to "torn surfaces" with large holes likely has an interpretation in terms of the appearance of a Higgs branch in the gauge theory when an m_α approaches an eigenvalue of Φ. It will be interesting to find a superpotential via which we can fix the moduli at a point on this Higgs branch.

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